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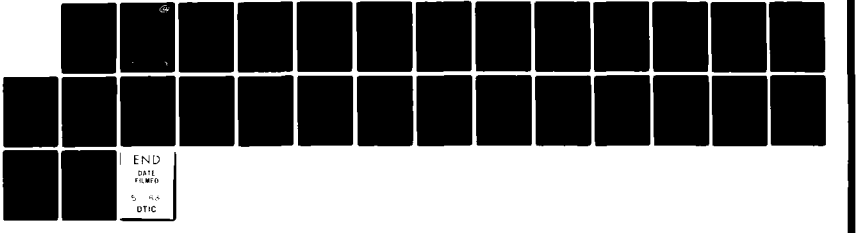

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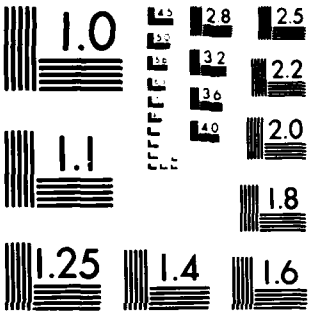


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A Technical Report

under Grant No. N00014-82-K-0409

# SPLINE FITTING THROUGH PARALLEL PROCESSING

Submitted to:

Dr. Lloyd Griffiths
Electronics Division
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

Submitted by:

D. Kazakos
Associate Professor

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April 1983

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UNIVERSITY OF VIRGINIA

**COMMUNICATIONS SYSTEMS LABORATORY**
**DEPARTMENT OF ELECTRICAL ENGINEERING**
**SCHOOL OF ENGINEERING AND APPLIED SCIENCES**
**UNIVERSITY OF VIRGINIA**

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A Technical Report

# SPLINE FITTING THROUGH PARALLEL PROCESSING

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Department of Electrical Engineering
RESEARCH LABORATORIES FOR THE ENGINEERING SCIENCES
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VIRGINIA

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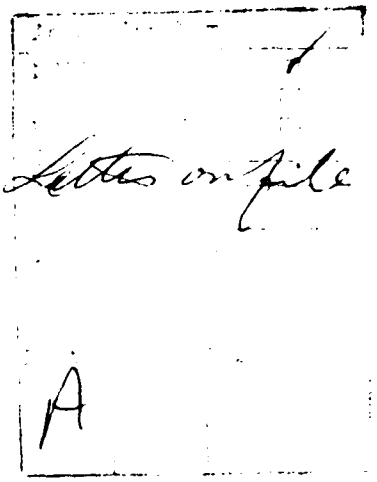
## ABSTRACT

A new method is presented for fitting polynomial splines to n equispaced data. Using Jain's [32] cyclic decomposition of banded Toeplitz matrices, we show that the operations can be performed by n-point Fast Fourier Transforms (FFT). Thus, the use of parallel processing FFT techniques provides a speed of $O(2\log_2 n)$, independently of the degree of the spline. Explicit solutions are derived for the cubic, quartic and quintic spline.

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## Introduction

In many applications there appears the need to represent a set of raw data by fitting to them a smooth function or set of functions. A great amount of theoretical work has been done within the framework of approximation theory in studying properties of curve fitting for various families of approximating functions.

One of the most attractive and well structured families of approximating functions are the splines. They have been extensively studied in the mathematical literature, for example in [1] - [7] and elsewhere.

Splines have been very useful in statistics. References [7] - [18] represent the most significant spline application papers in the statistical literature.

In the engineering literature, spline functions have been used as approximating tools, in the areas of Systems ([19] - [22]) and Pattern Recognition ([23] - [30]).

In ~~the present~~ paper we concentrate on a fast, parallel computation technique for fitting a spline to n equispaced data points. Existing techniques can fit splines in $0(n)$ time, with recursive (serial) processing of the data, as in [16]. The new technique is based on the use of Fast Fourier Transform, and its parallel processing capability. As a result, our technique achieves the fitting of a spline in $0(\log_2 n)$ time. The n data points must be equispaced.

## Cubic Spline Fit

Let $\{(x_i,y_i)\ \ i = 0, 1, \ldots, n\}$ be a set of data points with $a = x_0 < x_1 < \ldots < x_n = b$. We would like to fit to the data a function $S(x)$ that has two derivatives. We require that $S(x_i) = y_i$, $i = 0, 1, \ldots, n$ and that under the above requirement $S$ minimizes the integral

$$I_2(S) = \int_a^b [S''(x)]^2 dx- \tag{1}$$

under the conditions $S'(a) = S'(b) = 0$.

In the theory of spline functions it is shown [2] that the above constrained minimum is achieved when $S(x)$ is a set of piecewise cubic polynomials with continuous first and second derivatives at the points $\{x_i,\ i=1, \ldots, n=1\}$. We will be concerned here only with uniformly spaced $x_i$'s, hence $x_i = x_o + ih$, $i = 0, 1, \ldots, n$. $h$ = increment. The continuity requirement demands that:

$$S'(x_i^-) = S'(x_i^+), \quad S''(x_i^-) = S''(x_i^+) \tag{2}$$

for $i = 1, 2, \ldots, n-1$.

We denote $M_i$ the second derivative of $S(x)$ at $x_i$. Then

$$S''(x) = M_{i-1}(x_i-x)h^{-1} + M_i(x-x_{i-1})h^{-1},\ x_{i-1} \le x \le x_i \tag{3}$$

Integrating (3) twice and using the conditions $\{S(x_i) = y_i\}$ we obtain:

$$S(x) = M_{i-1}(x_i-x)^3(6h)^{-1} + M_i(x-x_{i-1})^3(6h)^{-1} +$$

$$+ (h^{-1}y_i - hM_i6^{-1})(x-x_{i-1}) + (h^{-1}y_{i-1} - hM_{i-1}6^{-1})(x_i-x),$$

$$x_{i-1} \le x \le x_i \tag{4}$$

Differentiating (4) once and using the continuity of the first derivative for i=1, ..., n-1, we find:

$$M_{i-1} + 4M_i + M_{i+1} = 6h^{-2}(y_{i-1} - 2y_i + y_{i+1}) \quad i=1, 2, \ldots, n-1 \tag{5}$$

We assume $M_o$ and $M_n$ are known. Then the set of unknowns is $M_1 M_2 \ldots M_{n-1}$, and we have a set of n-1 equations from (5) with an equal number of unknowns. In matrix notation the set of equations is:

$$TM = 6h^{-2}H \tag{6}$$

where

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ M_{n-1} \end{bmatrix}, \quad T = \begin{bmatrix} 4 & 1 & & & & & \\ 1 & 4 & 1 & & & 0 & \\ & 1 & 4 & 1 & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & 0 & & & \cdot & & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 4 \end{bmatrix} \tag{7}$$

$$H = \begin{bmatrix} y_0 = 2y_1 + y_2 - h^2M_0 6^{-1} \\ y_1 = 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n - h^2M_n 6^{-1} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \\ \\ h_{n-1} \end{bmatrix}$$

and T is an (n-1)x(n-1) matrix, H is an (n-1) vector. Clearly the basic problem here is the efficient inversion of T and the multiplication $T^{-1}H$. We define the following (n-1)x(n-1) matrix:

$$T_p = \begin{bmatrix} 1 & p & & & \\ p & 1 & p & 0 & \\ & p & 1 & p & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & p \\ & & & p & 1 \end{bmatrix}$$

It is a known result from [31] that the eigenvalues $q_k$ and eigenvectors $V_k$ of $T_p$ are:

$$q_k = 1 + 2p \cos(k\pi n^{-1}) \tag{8}$$

$$V_k = [\sin(k\pi n^{-1}), \sin(2k\pi n^{-1}), \ldots, \sin((n-1)k\pi n^{-1})]\sqrt{2/n}$$
$$k = 1, \ldots, n-1 \tag{9}$$

If we divide T by 4, we have the special case of (7) with p = 1/4. Using the eigenvector-eigenvalue expansion of T, we have:

$$T = 4 \sum_{k=1}^{n-1} [1 + 0.5 \cos(k\pi n^{-1})] V_k {}^t V_k \tag{10}$$

The inverse of T has the same eigenvectors and inverse eigenvalues. Therefore,

$$T^{-1} = 1/4 \sum_{k=1}^{n-1} [1 + 0.5 \cos(k\pi n^{-1})]^{-1} V_k {}^t V_k \tag{11}$$

The equation for the coefficient vector M is:

$$M = 1.5h^{-2} \sum_{k=1}^{n-1} s_k V_k {}^t V_k H \tag{12}$$

where

$$s_k = [1 + 0.5 \cos(k\pi n^{-1})]^{-1} \tag{13}$$

The matrix $V_k {}^t V_k$ has elements:

$$\{2n^{-1} \sin(km\pi n^{-1}),\ \sin(kq\pi n^{-1},\ m,\ q=1,\ \ldots,\ n-1\} \tag{14}$$

If we define $F = (f_1 \ \ldots \ f_{n-1})$,

$$f_k = n^{-1/2} \sum_{q=1}^{n-1} h_q \sin(kq\pi n^{-1}),\ k=1,\ \ldots,\ n-1 \tag{15}$$

then the mth component of the vector M is:

$$M_m = 3h^{-2} n^{-1/2} \sum_{k=1}^{n-1} s_k f_k \sin(km\pi n^{-1}) \tag{16}$$

Computations (15) and (16) are slightly modified versions of Finite Fourier Transforms, and each of them can be completed in $0(\log_2 n)$ time, if Fast Fourier Transform methods are used. In other words, F is produced from H by a Fast Fourier Transform, and then M is produced by the vector $(s_1 f_1, \ \ldots, \ s_{n-1} f_{n-1})$ by another Fast Fourier Transform. Hence the computation of M requires computation time of the order $0(\log_2 n)$.

The reason for obtaining such a simple solution is the fortunate event of the eigenvalues and eigenfunctions of T being expressed in a closed form that relates them to the Fast Fourier Transform. For higher order splines our solution requires to develop methods based on the approximation of Toeplitz matrices by circulant ones, to be presented in the next section. We will utilize a method of partitioning and cyclical decomposition of banded Toeplitz matrices, which is due to A.K. Jain [32].

## Circulant and Toeplitz Matrices

Part of the exposition in the present section follows Gray [33].

A circulant matrix C is one having the form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdot & \cdot & \cdot & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & \cdot & \cdot & c_{n-2} \\ \cdot & c_{n-1} & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & c_2 \\ \cdot & & & & & & c_1 \\ c_1 & \cdot & \cdot & \cdot & \cdot & c_{n-1} & c_0 \end{bmatrix} \tag{17}$$

The eigenvalues $q_m$ and eigenvectors $V_m$ of C are the solutions of

$$CV = qV, \quad V = [v_0, \ldots, v_{n-1}]^t \tag{18}$$

or equivalently of the n difference equations

$$\sum_{k=0}^{m-1} c_{n-m+k} v_k + \sum_{k=m}^{n-1} c_{k-m} v_k = q v_m, \quad m=0, 1, \ldots, n-1 \qquad (19)$$

It is easily verified for any m=0, 1, ..., n-1, that $v_k = \exp\{-2\pi imkn^{-1}\}$ is a solution to (19), resulting in the eigenvalues

$$q_m = \sum_{k=0}^{n-1} c_k \exp\{-2\pi imkn^{-1}\} \qquad (20)$$

$(i = \sqrt{-1})$

and the corresponding eigenvectors

$$V_m = n^{-1/2}[1, \exp(-2\pi imn^{-1}), \ldots, \exp(-2\pi im(n-1)n^{-1})]^t \qquad (21)$$

We can now write

$$C = n^{-1} \sum_{m=0}^{n-1} q_m V_m V_m^{*t} \qquad (22)$$

$$C^{-1} = n^{-1} \sum_{m=0}^{n-1} q_m^{-1} V_m V_m^{*t} \qquad (23)$$

We observe that all circulant matrices have the same set of eigenvectors. Also, the inverse of a circulant is also a circulant. The multiplication of $C^{-1}$ to any vector $H = (h_0 \ldots h_{n-1})^t$ can be done by Fast Fourier Transform techniques as follows:
Let $Z = (z_0 z_1 \ldots z_{n-1})^t$,

$$Z = C^{-1} H \qquad (24)$$

Let $f_k = n^{-1/2} \sum_{s=0}^{n-1} h_s \exp(-2\pi iskn^{-1})$ $\qquad (25)$

$k = 0, \ldots, n-1$

be the Finite Fourier Transform coefficients of H. Then, the components of Z,

$$z_m = n^{-1/2} \sum_{k=1}^{n-1} q_k^{-1} f_k \exp(2\pi i m k n^{-1}) \tag{26}$$

$$m = 0, \ldots, n-1$$

are Finite Fourier Transform coefficients. Hence, Z is computable from (26) in $0(\log_2 n)$ computation time. A Toeplitz matrix $T_n$ of order (n, m, s), with $s < n$, $m < n$ is defined as an nxn matrix with entries t(k, j) such that

$$t(k, j) = \begin{cases} t(k-j) & \text{for } -s \leq k-j \leq m \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

and $t(m) \neq 0$, $t(-s) \neq 0$.

$$T_n = \begin{bmatrix} t(0) & t(-1) & \cdots & t(-s) & & 0 \\ \vdots & t(0) & & & \ddots & \\ & & \ddots & & & \\ & & & \ddots & & \\ t(m) & & & t(0) & \cdots & t(-s) \\ & \ddots & & & \ddots & \vdots \\ 0 & & & t(m) & \cdots & t(0) \end{bmatrix} \tag{28}$$

With the exception of the upper right and lower left corners $T_n$ looks like a criculant matrix; i.e., each row is the row above it shifted to the right one place. If we fill in the upper right and lower left corners by the appropriate entries, we can make $T_n$ exactly a circulant. Define the circulant matrix C in this way

$$c_k = \begin{cases} t(-k), & k = 0, ., \ldots, s \\ t(n-k), & k = n-m, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

C, defined as above, is a prime candidate for approximating $T_n$. The rationale for such an approximation is the desire to approximate any operation $T_n^{-1}Y$, Y = vector, by the operation $C_n^{-1}Y$, which can be performed in $0(\log_2 n)$ computation time using the Fast Fourier Transform.

Let D be the difference

$$D = C - T_n \tag{30}$$

D has the form

$$D = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix} \quad \text{where } Q = \begin{bmatrix} t(m) & t(m-1) & \cdots & t(1) \\ & t(m) & & t(m-1) \\ & 0 & \ddots & \\ & & & t(m) \end{bmatrix}$$

$$P = \begin{bmatrix} t(-s) & & 0 \\ t(1-s) & \ddots & \\ \vdots & \ddots & \ddots \\ t(-1) & \cdots & t(1-s) \quad t(-s) \end{bmatrix}$$

Now suppose that we want to solve the equation $T_n X = Y$, where X, Y are n-vectors. Substitute $T_n = C - D$. Then we have

$$CX = DX + Y \quad \text{or} \quad X = C^{-1}DX + C^{-1}Y \tag{31}$$

We partition X and $C^{-1}Y$ as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \quad C^{-1}Y = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \tag{32}$$

where $X_1$, $W_1$ have dimension s, $W_3$, $X_3$ have dimension m.

We also partition $C^{-1}$

$$C^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{33}$$

The dimensions of $A_{ij}$ will become obvious from the next equations. Equation (31) becomes

$$X_1 = A_{11}QX_3 + A_{13}PX_1 + W_1 \tag{34}$$

$$X_2 = A_{21}QX_3 + A_{23}PX_1 + W_2 \tag{35}$$

$$X_3 = A_{31}QX_3 + A_{33}PX_1 + W_3 \tag{36}$$

Now, we can solve (34) and (36) for $(X_1, X_3)$ and then substitute them in (35) to find $X_2$. Solution:

$$\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = \left[ I - \begin{bmatrix} A_{13}P, & A_{11}Q \\ A_{33}P, & A_{31}Q \end{bmatrix} \right]^{-1} \begin{bmatrix} W_1 \\ W_3 \end{bmatrix} \tag{37}$$

The method of solution of equations (34) - (36) through partitioning and cyclical decomposition of the banded Toeplitz matrices, is due to A.K. Jain [32] and is a very efficient method for solving a system of linear equations when the matrix $T_n$ is of the banded Toeplitz form.

Using parallel processing techniques through Fast Fourier Transform architecture, the solution of equations (34) - (37) is achieved in $(s+m)^3 + 0(2\log_2 n)$ time.

## Quartic and Quintic Spline Fit

In the present section we will use the method of approximating a Toeplitz matrix by a circulant to solve efficiently the spline fit problem for higher degree splines. Explicit solutions will be given the quartic and quintic spline, which are the simplest and hence more useful higher degree splines. To avoid proliferation of notation, we will use $M_j$ to denote the kth derivative at the knot $x_j$ of a k+1 degree spline. Hence, k=2, 3, 4 correspond to the cubic, quartic and quintic spline. The number of knots used will be n+k for the k+1 degree spline, so that k of them serve as boundary points with parameters assumed known, leaving exactly n unknown parameters. They form the vector $M = [M_1 \ldots M_n]^t$. Hence the matrix to be inverted will be always nxn. The vector M of kth derivatives together with the continuity of the first k derivatives are sufficient to define the spline completely.

We consider now the quartic spline. Let

$$I_4 = \{x_j = jh,\ a = -h,\ b = (n+1)h,\ j = -1, 0, 1, \ldots, n, n+1\} \tag{38}$$

be a set of n+3 equispaced knots on [a, b]. A quartic spline S(x) defined on [a, b] is a piecewise fourth degree polynomial between knots $(x_j x_{j+1})$ that fits a set of data $\{S(x_j) = y_j,\ j\varepsilon I_4\}$ and has continuous the first three derivatives at the knots, i.e.

$$S^{(k)}(x_j^-) = S^{(k)}(x_j^+),\quad k = 0, 1, 2, 3,\ j\varepsilon I_4 \tag{39}$$

$$\text{Let } M_j = S^{(3)}(x_j^-) = S^{(3)}(x_j^+),\quad j\varepsilon I_4 \tag{40}$$

We have (n+3) parameters $\{ M_j,\ j \in I_4 \}$. The quartic spline requires specification of the following 3 boundary numbers:

$S'(a),\ S''(a),\ S'(b),$

It is shown in [4] that the 3 boundary conditions specify uniquely the "boundary parameters" $M_{-1},\ M_0,\ M_{n+1}$. Therefore, we will consider them known. It is further shown in [4] that $M_j$ satisfy the following equations:

$$M_{k-2} + 11M_{k-1} + 11M_k + M_{k+1} = 24h^{-3}(-y_{k-2} + 3y_{k-1} - 3y_k + y_{k+1})$$
$$\text{for } k = 1, 2, \ldots, n \tag{41}$$

We have now n equations and an equal number of unknown parameters, constituting the vector $M = [M_1 \ldots M_n]^t$.

We define the n constants

$$f_k = \begin{bmatrix} -h^3M_0 \cdot 11/24 - h^3M_{-1}/24 + (y_2 - 3y_1 + 3y_0 - y_{-1}), & k=1 \\ -h^3M_0/24 + (y_3 - 3y_2 + 3y_1 - y_0), & k=2 \\ \vdots & \\ (y_{j+1} - 3y_j + 3y_{j-1} - y_{j-2}), & 3 \le k = j \le n-1 \\ \vdots & \\ h^3M_{n+1}/24 + (y_{n+1} - 3y_n + 3y_{n-1} - y_{n-2}), & k=n \end{bmatrix} \tag{42}$$

Let $F = (f_1 \ldots f_n)^t$

We also define the nxn Toeplitz matrix

(13)

$$T = \begin{bmatrix} b & 1 & & 0 & & \\ b & b & 1 & & & \\ 1 & b & & & & \\ & 1 & & & & \\ 0 & & & b & b & 1 \\ & & & 1 & b & b \end{bmatrix}, \quad b = 11 \tag{43}$$

Then, the vector M from the solution of equations (43) is:

$$M = 24h^{-3}T^{-1}F \tag{44}$$

At this point, we are ready to apply directly the procedure of Section II because T is banded Toeplitz. If we identify T with $T_n$ from eq. (30), we have: s = 1, m = 2, t(0) = t(1) = b, t(2) = 1, t(-1) = 1. The matrices P, Q are:

$$Q = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad P = 1 \quad \text{(scalar)} \tag{45}$$

Also,

$$Y = 24h^{-3}F, \quad x_1 = \text{scalar}$$

$x_3$ = two dimensional column vector. Here the unknown vector M is identified with X. Using equations (34) - (37), the solution for M=X is immediate. The time required for the operations is $3^3 + 0(2 \log_2 n)$.

Finally, we will derive the solution to the quintic spline fit problem. Let

$$I_5 = \{x_j = jh,\ j = -1, 0, 1, \ldots, n, n+1, n+2,\ a = -h,\ b = (n+2)h\} \tag{46}$$

be a set of (n+4) equispaced knots in [a, b]. A quintic spline S(x) defined on [a, b] is a function that has continuous the three first derivatives on [a, b], fits a set of data $y_j = S(y_j)$, $j \in I_5$, and

minimizes the integral

$$\int_a^b [S^{(3)}(x)]^2 dx \tag{47}$$

under 4 specified boundary conditions:

$$S'(a),\ S''(a),\ S'(b),\ S''(b).$$

(The quintic spline corresponds to a differential operator $L=D^3$). In spline theory [4] it is shown that the solution to the above constrained minimization problem is a piecewise fifth order polynomial with the first four derivatives continuous at the joints. Hence,

$$S(x_j) = y_j,\ S^{(k)}(x_j^-) = S^{(k)}(x_j^+),\ k = 0, 1, 2, 3, 4,\ j\varepsilon I_5 \tag{48}$$

Let $$M_j = S^{(4)}(x_j^-) = S^{(4)}(x_j^+),\ j\varepsilon I_5 \tag{49}$$

The 4 boundary conditions specify uniquely the 4 "boundary parameters" $M_{-1}$, $M_0$, $M_{n+1}$, $M_{n+1}$, as shown in [4]. Therefore, we will consider them known. It is shown in [4] that the following set of n equations is satisfied by the $M_j$'s:

$$M_{k-2} + 26M_{k-1} + 66M_k + 26M_{k+1} + M_{k+2} =$$

$$120h^{-4}[y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}]$$

$$\text{for } k = 1, 2, \ldots, n \tag{50}$$

Now we have n equations and n unknown parameters, the components of the vector $M = [M_1 M_2 \ldots M_n]$. Let $F = [f_1 f_2 \ldots f_n]$, where:

(15)

$$f_k = \begin{bmatrix} (y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3) - 26 \cdot 120^{-1}h^4M_0 - \\ - 120^{-1}h^4M_{-1} \quad \text{for } k=1 \\ (y_0 - 4y_1 + 6y_2 - 4y_3 + y_4) - 120^{-1}h^4M_0 \quad \text{for } k=2 \\ \vdots \\ (y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}), \text{ for } 3 \le k \le n-2 \\ (y_{n-3} - 4y_{n-2} + 6y_{n-1} - 4y_n + y_{n+1}) - 120^{-1}h^4M_{n+1}, \\ \text{for } k = n-1 \\ (y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}) - 26.120^{-1}h^4M_{n+1} - \\ - h^4120^{-1}M_{n+2}, \quad \text{for } k=n \end{bmatrix} \tag{51}$$

Let also

$$T = \begin{bmatrix} 66 & 26 & 1 & & & & & \\ 26 & 66 & 26 & & & 0 & & \\ 1 & 26 & & & & & & \\ & & & & & & & \\ & & & & & & & 1 \\ & & & & & & & \\ 0 & & & & & & & \\ & & & & & & 66 & 26 \\ & & & & & 1 & 26 & 66 \end{bmatrix} \tag{52}$$

Then the set of equations (52) becomes:

$$TM = 120^{-4}F \tag{53}$$

Identifying the parameters of the present problem to the previous results of Section II, we have: s=m=2

(16)

$$P = \begin{bmatrix} 1 & 0 \\ 26 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 26 \\ 0 & 1 \end{bmatrix} \tag{54}$$

$$X_1 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \qquad X_3 = \begin{bmatrix} M_{n-1} \\ M_n \end{bmatrix}$$

$$Y = 120h^{-4}F \tag{55}$$

Using (39) we can compute $X_1$, $X_3$ with $4^3$ operations. Combining with (37) we compute $X_2$. The total computational time required is: $4^3 + 0(2\log_2 n)$.

Generally, to fit a k+1 degree polynomial spline to n+k equispaced knots with data $\{y_j\}$, we have n+k-1 intervals and n+k-2 interior points. For each interval, specification of the spline requires k+2 constants to be solved for. Hence the number of unknown parameters is (k+2)(n+k-1). The continuity of $S^{(j)}(x)$, j = 0, 1, ..., k at the interior knots (joints) provides (k+1)·(n+k-2) equations. By fitting the data points $\{y_j\}$ we get (n+k) equations. Let s be the number of specified boundary conditions. In order to have equal number of unknown parameters and equations, we must have:

$$(k+2)(n+k-1) = (k+1)(n+k-2) + (n+k) + 2$$

hence

$$s=k.$$

The boundary conditions can be spec i fied either by a number of derivatives at the end points a, b or by a number of boundary values of the moments $M_j$. For the cubic spline, k=2, and we picked $M_0$, $M_{n+1}$ as the known boundary values. For the quartic

spline, we have k=3. The moments ($M_{-1}$, $M_0$, $M_{n+1}$) were assumed known. For the quintic, we have k=4 and we used the moments ($M_{-1}$, $M_0$, $M_{n+1}$, $M_{n+2}$) as known boundary conditions. An alternate set of boundary conditions that has been frequently used is the specification of a total of k derivatives of S(x) at a and b. There is a one to one dependence between the two different boundary conditions [4]. In the present paper, the boundary moments were chosen as more convenient.

The kth derivative of a (k+1) degree spline is a piecewise linear function, expressed as:

$$S^{(k)}(x) = M_{j-1}h^{-1}(x_j - x) + M_j h^{-1}(x-x_{j-1}) \text{ for } x_{j-1} \leq x \leq x_j \tag{56}$$

where

$$M_j = S^{(k)}(x_j)$$

Up to this point we have shown that the parameters $\{M_j\}$, which are the third derivatives at $\{x_j\}$ for the quartic spline and fourth derivatives at $x_j$ for the quintic spline, can be computed from the data $y_j$ in parallel processing time of $2\log_2 n$ operations.

Complete specification of the other parameters of the spline will be now undertaken.

For the quartic spline, we have:

$$S^{(3)}(x) = M_{j-1}h^{-1}(x_j - x) + M_j h^{-1}(x - x_{j-1})$$

$$\text{for } x_{j-1} \leq x \leq x_j \tag{57}$$

Integrating three times, we have:

$$S(x) = -(24h)^{-1} M_{j-1}(x_j - x)^4 + (24h)^{-1} M_j(x - x_{j-1})^4 +$$

$$+ 2^{-1}hB_j^1(x-x_j)^2 + h^2B_j^2(x - x_j) + h^3B_j^3, \quad x_{j-1} \le x \le x_j \qquad (58)$$

where $\{B_j^1, B_j^2, B_j^3\}$ are constants that need to be determined from $\{M_j\}$, $\{y_j\}$, and the continuity requirement at the joints. We take the consecutive derivatives of $S(x)$, write the corresponding expressions of $S^{(k)}(x)$ for the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, then apply the continuity requirement for $S^{(k)}(x)$ at $x = x_j$, for $k = 1, 2$. The following equations result:

$$S^{(2)}(x_j^-) = S^{(2)}(x_j^+); \quad B_{j+1}^1 = B_j^1 + M_j, \quad j = 0, 1, \ldots, n \qquad (59)$$

$$S^{(1)}(x_j^-) = S^{(1)}(x_j^+); \quad B_{j+1}^2 = B_j^2 + B_{j+1}^1, \quad j = 0, 1, \ldots, n \qquad (60)$$

The equations $\{S(x_j^-) = y_j\}$ provide the following relationships:

$$y_j = S(x_j^-); \quad B_j^3 = h^{-3}y_j - 24^{-1}M_j, \quad j = 0, 1, \ldots, n, n+1 \qquad (61)$$

$$S(a^+) = y_{-1}; \quad 2^{-1}B_0^1 - B_0^2 = -B_0^3 + h^{-3}y_{-1} + 24^{-1}M_{-1} \qquad (62)$$

We still need to determine the two initial values, $(B_0^1, B_0^2)$. For this purpose, we use one additional equation:

$$S(x_0^+) = y_0; \quad h^{-3}y_0 = -24^{-1}M_0 + 2^{-1}B_1^1 - B_1^2 + B_1^3 \qquad (63)$$

Substituting $(B_1^1, B_1^2)$ from (59), (60) for j=0, in terms of $B_0^1, B_0^2)$, we get:

$$2^{-1}B_0^1 + B_0^2 = B_1^3 - M_0 .13/24 - h^{-3}y_0 \qquad (64)$$

From (62) and (64) we can solve for $(B_0^1, B_0^2)$. Then $B_j^1, B_j^2, B_j^3$ are computable from $M_j$.

For the quintic spline, we have:

$$S^{(4)}(x) = M_{j-1}\, h^{-1}(x_j - x) + M_j h^{-1}(x - x_{j-1})$$

$$\text{for } x_{j-1} \le x \le x_j \tag{65}$$

Integrating four times, we find:

$$S(x) = (120h)^{-1} M_{j-1}(x_j - x)^5 + (120h)^{-1} M_j(x - x_{j-1})^5 +$$

$$+ 6^{-1}h\, B_j^1(x - x_j)^3 + 2^{-1}h^2 B_j^2(x - x_j)^2 + h^3\, B_j^3(x - x_j)$$

$$+ h^4\, B_j^4 \tag{66}$$

$$\text{for } x_{j-1} \le x \le x_j$$

The constants now are $B_j^1$, $B_j^2$, $B_j^3$, $B_j^4$ . We take the consecutive derivatives $S^{(k)}(x)$ for the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, then apply the continuity requirement for $S^{(k)}(x)$ at $x = x_j$, $k = 1, 2, 3$. We also apply the condition $S(x_j^-) = y_j$. The resulting equations are:

$$S(x_j^-) = y_j; \quad B_j^4 = h^{-4}y_j - 120^{-1}M_j \quad j = 0, 1, \ldots, n+2 \tag{67}$$

$$S^{(3)}(x_j^-) = S^{(3)}(x_j^+); \quad B_{j+1}^1 = B_j^1 + M_j \tag{68}$$

$$S^{(2)}(x_j^-) = S^{(2)}(x_j^+); \quad B_{j+1}^2 = B_{j+1}^1 + B_j^2 \tag{69}$$

$$S^{(1)}(x_j^-) = S^{(1)}(x_j^+); \quad B_{j+1}^3 = B_j^3 + B_{j+1}^2 - 2^{-1}\, B_{j+1}^1 + 12^{-1}M_j \tag{70}$$

$$\text{for } j = 0, 1, \ldots, n+1$$

We need now to solve for the initial values $(B_0^1, B_0^2, B_0^3)$, which, together with the equations (67) - (70), will give the complete solution. For this purpose, we will formulate a set of 3 equations, in which the only unknown parameters are $(B_0^1, B_0^2, B_0^3)$.

Note that $B_j^4$ are immediately found from (67). From (66) we have:

$$h^{-4}.S(x_{j-1}^+) = 120^{-1}M_{j-1} - 6^{-1}B_j^1 + 2^{-1}B^2 - B_j^3 + B_j^4 \qquad (71)$$

for j = 0, 1, ..., n+2

We apply (71) for j = 0, 1, 2. The result is:

$$y_{-1}h^{-4} = 120^{-1}M_{-1} - 6^{-1}B_0^1 + 2^{-1}B_0^2 - B_0^3 + B_0^4 \qquad (72)$$

$$y_0h^{-4} = 120^{-1}M_0 - 6^{-1}B_1^1 + 2^{-1}B_1^2 - B_1^3 + B_1^4 \qquad (73)$$

$$y_1h^{-4} = 120^{-1}M_1 - 6^{-1}B_2^1 + 2^{-1}B_2^2 - B_2^3 + B_2^4 \qquad (74)$$

Using (68) - (70), we express $(B_1^1, B_1^2, B_1^3, B_2^1, B_2^2, B_2^3)$ in terms of $(B_0^1, B_0^2, B_0^3)$ and substitute them in (72) - (74). Thus we finally have 3 equations with the unknown parameters $(B_0^1, B_0^2, B_0^3)$. After the determination of the initial values $B_0^k$, equations (68) - (69) provide all the parameters $B_j^k$.

## Conclusions

We have presented an efficient algorithm for fitting a polynomial spline to a set of n equidistant data. The algorithm exploits the parallel nature of Fast Fourier Transform and the form of the difference equations relating the moments $\{M_j\}$ to the data $\{y_j\}$. Explicit formulas were obtained for the cubic, quartic and quintic spline, and the method is readily extensible to higher order and more general splines. The method involves only linear operations, and is much simpler than previous ones. The time complexity of the method is $0(2\log_2 n)$, independently of the order of the spline.

## References


[1] I.J. Schoenberg, "Contributions to the problem of approximation of equidistant data by analytic functions", Quart. Appl. Math. 4, (1946), part A, pp. 45-99; part B, pp. 112-141.

[2] G. Birkhoff, "Piecewise Bicubic Interpolation and Approximation in Polygons", in Approximations with Special Emphasis on Spline Functions, pp. 185-221, I.J. Schoenberg, editor, Academic Press, New York, 1969.

[3] C. de Boor and R.E. Lynch, "On Splines and Their Minimum Properties", J. Math. Mech., Vol. 15, (1966), pp. 953-969.

[4] J.H. Ahlberg, E.N. Nilson and J.L. Walsh, The Theory of Splines and Their Applications, Academic Press, New York, 1967.

[5] J. Jerome and L. Schumaker, "On Lg-Splines", J. Approx. Theory, Vol. 2, 1969, pp. 29-49.

[6] T.N.E. Greville (ed.), Theory and Applications of Spline Functions, Academic Press, New York, 1969.

[7] C. de Boor, "On Calculating with B-Splines", Journal of Approx. Theo., Vol. 6, 1972, pp. 50-62.

[8] G. Kimeldorf and G. Wahba, "A Correspondence Between Bayesian Estimation on Stochastic Processes and Smoothing by Splines", Ann. Math. Stat., Vol. 41, 1970, pp. 495-502.

[9] G. Kimeldorf and G. Wahba, "Spline Functions and Stochastic Processes", Sankhya, Vol. 132, pp. 173-183.

[10] G. Kimeldorf and G. Wahba, "Some Results on Tchebycheffian Spline Functions", J. Math. Anal. Appl., Vol. 33, 1971, pp. 82-95.

[11] G. Wahba, "A Polynomial Algorithm for Density Estimation", Ann. Math. Statis., Vol. 42, 1971, pp. 1870-1886.

[12] G. Wahba, "Interpolating Spline Methods for Density Estimation I. Equispaced Knots", Ann. Statist., Vol. 3, pp. 30-48, 1975.

[13] I.W. Wright and E.J. Wegman, "Isotonic, Convex and Related Splines", Annals of Stat., vol. 8, no. 5, 1980, pp. 1023-1035.

[14] E.J. Wegman, "Vector Splines and the Estimation of Filter Functions", Technometrics, vol. 23, no. 1, Feb. 1981, pp. 83-89.

[15] E.J. Wegman, "Two Approaches to Nonparametric Regression: Splines and Isotonic Inference", in Recent Developments in Statistical Inference and Data Analysis, K. Matusita, editor, North Holland Publishing Co., 1980.

[16] H.L. Weinert and T. Kailath, "Stochastic Interpretations and Recursive Algorithms for Spline Functions", Ann. Statist., Vol. 2, 1974, pp. 787-794.

[17] P.J. Huber, "Fisher Information and Spline Interpolation", Ann. of Statist. Vol. 2, 1974, pp. 1029-1033.

[18] G.F. de Montricher, R.A. Tapia and J.R. Thompson, "Nonparametric Maximum Likelihood Estimation of Probability Densities by Penalty Function Methods", Ann. Statis. Vol. 3, pp. 1329-1348, 1975.

[19] R.J.P. de Figueiredo and A.N. Netravali, "Optimal Spline Digital Simulators of Analog Filters", edited by I.W. Sandberg and J.F. Kaiser, of the IEEE Trans. on Circuit Theory, Vol. CT-18, pp. 711-717, 1971.

[20] D.G. Lainiotis and J.G. Deshpande, "Parameter Estimation Using Splines", J. Inform. Sciences, Vol. 7, No. 3, pp. 101-125, 1974.

[21] O. Mangasarian and L. Shumaker, "Splines via Optimal Control", in Approximations with Special Emphasis on Spline Functions, I. Schoenberg, Editor, New York; Academic Press, 1969, pp. 119-156.

[22] H.L. Weinert and T. Kailath, "A Spline Theoretic Approach to Minimum Energy Control", IEEE Trans. on Aut. Control, Vol. AC-21, pp. 391-393, June 1976.

[23] T. Pavlidis and G.S. Fang, "A Segmentation Technique for Waveform Classification", IEEE Trans. on Computers, Vol. C-21, pp. 901-904, 1972.

[24] T. Pavlidis, "Waveform Segmentation Through Functional Approximation", IEEE Trans. on Comp. Vol. C-22, No. 7, pp. 689-607, 1973.

[25] T. Pavlidis and S.L. Horowitz, "Segmentation of Plane Curves", IEEE Trans. on Computers, Vol. C-23, No. 8, Aug. 1974, pp. 860-870.

[26] T. Pavlidis and A.P. Maika, "Uniform Piecewise Polynomial Approximation With Variable Joints", J. Approximation Theory, Vol. 12, Sept. 1974, pp. 61-69.

[27] T. Pavlidis, "Optimal Piecewise Polynomial $L_2$ Approximation of Functions of One and Two Variables", IEEE Trans. on Computers, Vol. C-24, Jan. 1975, pp. 98-102.

[28] T. Pavlidis, "An Algorithm for Continuous Piecewise Linear Approximations", Proceedings of the 1975 Johns Hopkins Conference on Information Sciences and Systems, pp. 9-14.

[29] R.L.P. Chang and T. Pavlidis, "Functional Approximation with Variable-Knot, Variable-Degree Splines", Tech. Report, No. 1, Computer Science Laboratory, Dept. of Electrical Engineering, Princeton University, Princeton, N.J., February 1976.

[30] T. Pavlidis, "Syntactic Pattern Recognition on the Basis of Functional Approximation", Conference Record of the 1976 Joint Workshop on Pattern Recognition and Artificial Intelligence, Hyannis, Mass, June 1-3, 1976, IEEE Publication No.: 76CH1169-2C.

[31] U. Grenander and G. Szego, Toeplitz Forms and Their Applications, University of California Press, Berkeley and Los Angeles, CA, 1958.

[32] A.K. Jain, "Fast Inversion of Banded Toeplitz Matrices by Circular Decomposition", IEEE Trans. on Acoustics, Speech and Signal Processing, vol. 26, no. 2, pp. 121-126, April 1978.

[33] R.M. Gray, "On the Asymptotic Eigenvalue Distribution of Toeplitz Matrices", IEEE Trans. on Info. Theory, Vol. IT-18, No. 6, Nov. 1972, pp. 725-730.

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